INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 40, Northern Autumn 2018 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Let ABC be a right-angled triangle with $\angle B = 90°$. A circle through B and the midpoint K of hypotenuse AC intersects the sides AB and BC at points M and N respectively. Suppose that $AC = 2MN$. Prove that M and N are the midpoints of the sides AB and BC respectively. (4 points)

2. Determine all positive integers n such that the numbers $1, 2, \ldots, 2n$ can be divided into pairs so that the product of sums of the numbers in each pair is a perfect square. (4 points)

3. A grid rectangle of the size 7×14 is divided along the grid lines into the squares of the size 2×2 consisting of 4 squares and corners consisting of 3 squares. Is it possible that the number of squares of the size 2×2 is

(a) equal to the number of corners? (1 point)

(b) greater than the number of corners? (3 points)

4. Nastya has 5 coins, which look identical, three of which are real and of the same weight, and the other two are fake. Of the fakes, one weighs more than a real coin, and the other weighs less than a real coin by the same amount. Nastya can ask an expert to perform three weighings of her choice on a simple balance. Then the expert reports the results to Nastya. Note that the results of all weighings are reported to Nastya after the third weighing. Could Nastya choose the weighings so that she would be able to determine both fake coins and state which of them is heavier for sure? A simple balance shows which of two sides is heavier/higher or if they are balanced. (5 points)

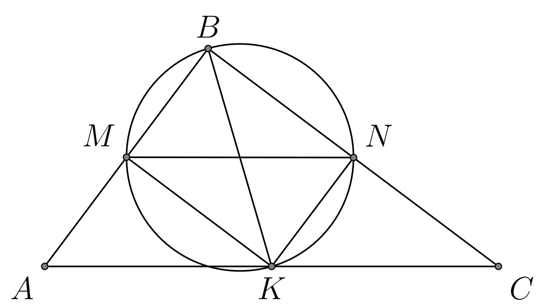
5. A nine-digit integer is called *beautiful* if all of its digits are different. Prove that there exist at least 1000 beautiful numbers, each of which is divisible by 37. (5 points)

O Level Junior Paper Solutions

Prepared by Oleksiy Yevdokimov and Greg Gamble

1. Let Γ be the circle through B and the midpoint K of hypotenuse AC. Since Γ passes through M and N, and $90^\circ = \angle ABC = \angle MBN$, MN is a diameter of Γ.

   Since K is the midpoint of hypotenuse AC, K is the circumcentre of triangle ABC. Thus, we have
   $$BK = \frac{1}{2}AC = MN,$$
   so that BK is also a diameter. Therefore, $\angle BMK = 90^\circ$, and hence KM is parallel to BC. Since K is the midpoint of AC, it follows that KM is a middle line of triangle ABC, so that M is the midpoint of AB. Similarly, KN is also a middle line of triangle ABC, and N is the midpoint of BC.



2. Let $\mathcal{S}$ be the chosen partition of $\{1, 2, \ldots, 2n\}$ into pairs, and P be its corresponding product of pair-sums.

   Solution 1. There is a partition $\mathcal{S}$ for which P is a perfect square, for any $n > 1$.

   For $n = 1$, there is only one choice for $\mathcal{S}$, namely $\{\{1, 2\}\}$, and hence necessarily $P = 1 + 2 = 3$, which is not a perfect square.

   For even n, we have $n = 2k$ for some integer $k \geq 1$, and we can choose
   $$\begin{aligned}&\mathcal{S} = \{\{1, 2n\},\ \{2, 2n-1\}, \ldots, \{n, n+1\}\},\ \text{giving}\\ &P = (1+2n)(2+(2n-1))\cdots(n+(n+1)) = (2n+1)^n = ((2n+1)^k)^2,\end{aligned}$$
   a perfect square.

   For $n = 3$, we can choose $\mathcal{S} = \{\{1, 5\}\ \{2, 4\},\ \{3, 6\}\}$, giving $P = 6 \cdot 6 \cdot 9 = 6^2 \cdot 3^2$, a perfect square.

   For odd $n > 3$, we have $n = 2k + 1$ for some integer $k > 1$. The key idea is that $n - 3$ is even, so that after partitioning $\{1, 2, \ldots, 6\}$ into pairs as we did for $n = 3$, the remaining $n - 3$ numbers can be partitioned into pairs in the way we did for even n, i.e. we can choose $\mathcal{S}$ and hence P as
   $$\begin{aligned}\mathcal{S} &= \{\{1, 5\},\ \{2, 4\},\ \{3, 6\}\ \{7, 2n\},\ \{8, 2n-1\}\ \{n+3, n+4\}\},\\ P &= 6 \cdot 6 \cdot 9 \cdot (7+2n)(8+(2n-1))\cdots((n+3)+(n+4))\\ &= 6^2 \cdot 3^2 \cdot (2n+7)^{n-3} = (18(2n+7)^{k-1})^2,\end{aligned}$$
   where P is again a perfect square.

Solution 2. There is a partition $\mathcal{S}$ for which P is a perfect square, for any $n > 1$.

The key idea is that any four consecutive integers $a, a+1, a+2, a+3$ can be partitioned into the pairs $\{a, a+3\}$, $\{a+1, a+2\}$ whose contribution to P is a square, namely

$$(a + (a+3))((a+1) + (a+2)) = (2a+3)^2.$$

Thus, if $2n$ is divisible by 4, i.e. n is even, we first partition $\{1, 2, \ldots, 2n\}$ into consecutive sets of 4, and then partition the sets of four as pairs, leading to P as a product of squares.

If $2n = 2$, i.e. $n = 1$, as we saw in Solution 1., $P = 3$ is unique and not a square.

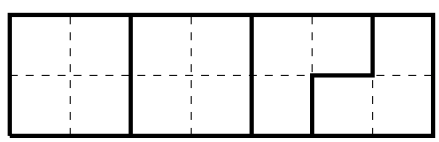
This leaves $2n$ is not divisible by 4 having remainder 2, for which $2n \geq 6$. After setting aside the first six numbers, what remains can first be partitioned into consecutive sets of 4. The first 6 can be partitioned into pairs as per Solution 1., giving a contribution of 18^2 to P, and by the strategy above each set of 4 can be partitioned into pairs that give a contribution to P that is a square.

Thus, for all $n > 1$, there is a partition $\mathcal{S}$ for which P is a product of squares, so that as a consequence P is a perfect square.

Note. For $n = 2$ and $n = 3$ the partitions $\mathcal{S}$ that give a square P are unique. For any other n they are not unique.

3. Let s and c be the numbers of 2×2 *squares* and *corners*, respectively.

   (a) Yes, it is possible to partition the 7×14 grid into equal numbers of 2×2 *squares* and *corners*. We note that orientation is irrelevant and for convenience have shown a 7×2 grid, with two rows and seven columns, with 2 *squares* and 2 *corners*.

   With 7 such 7×2 grids we have $c = s = 14$, as required.

   (b) **Solution 1.** No, $s > c$ is not possible. We will show c is at least 14. Orienting as in (a), we have 14 rows and 7 columns. Since the number of cells in each row is 7, which is odd and a 2×2 *square* only contributes an even number of cells to a row, we must have a *corner* in each row. However, if a *corner* contributes one cell to one row, then to an adjacent row it contributes two cells. Thus we need at least as many corners as rows, i.e. we need 14 corners, but then at most $s = (7 \times 14 - 3 \times 14)/4 = 14 \not> c$ (a contradiction). So $s > c$ is not possible.

   Solution 2. No, $s > c$ is not possible. Note that the total number of cells of the grid rectangle is $7 \times 14 = 98$. Orienting as in (a), we have 14 rows and 7 columns. Colour the columns black and white alternately, with the first column coloured black. Then 4 columns comprising a total of 56 cells are black and 3 columns comprising 42 cells are white. Thus there are 14 more black cells than white. A 2×2 square must cover 2 black cells and 2 white cells, whereas a corner must cover 2 cells of one colour and one of the other colour. Thus to make up the difference there must be at least 14 corners each covering an extra black cell. But then at most $s = (7 \times 14 - 3 \times 14)/4 = 14 \not> c$ (a contradiction). So $s > c$ is not possible.

4. **Solution 1.** Yes, Nastya can choose the weighings so that she can assuredly determine both fake coins and state which of them is heavier. Denote the five coins by a, b, c, d and e. Nastya can achieve the desired outcome by demanding the following three weighings:

   (1) a vs b

   (2) c vs d

   (3) a, b vs c, d.

   Noting that having omitted e from the weighings, at least one of a, b, c, d is fake, and so at least one of (1) and (2) is unbalanced. By relabelling the coins and weighings (1) and (2), if necessary, we need only consider the following 4 cases (where we identify a coin with its weight).

   Case 1: Both (1) and (2) are unbalanced; without loss of generality:

   $$a > b,\ c > d,\ a + b > c + d.$$

   Since (1) and (2) are unbalanced there is a fake coin in each weighing. So e is real. Also, (3) shows the heavy fake is one of a and b, and so (1) shows that a is the heavy fake. Similarly, (3) shows the light fake is one of c and d, and so (2) shows the light fake is d.

   This leaves cases where only one of (1) and (2) is unbalanced; without loss of generality (2) is unbalanced with $c > d$. Consequently, with (1) balanced, a and b are real. Then we must consider all possibilities for (3):

   Case 2: $a = b$, $c > d$, $a + b = c + d$. Since $c + d$ balances $a + b$ (with a and b both real), one of c and d is a heavy fake coin and the other a light fake coin. Hence (2) shows that c is the heavy fake coin, and d the light fake coin.

   Case 3: $a = b$, $c > d$, $a + b > c + d$. Since a and b both real, and (3) is unbalanced only one of c and d is fake, and since $a + b$ is the heavy side, one of c and d is a light fake. Hence, since (2) has a real coin on one side, the light fake must be d and the heavy fake is e.

   Case 4: $a = b$, $c > d$, $a + b < c + d$. Since a and b both real, and (3) is unbalanced only one of c and d is fake, and since $a + b$ is the light side, one of c and d is a heavy fake. Hence, since (2) has a real coin on one side, the heavy fake must be c and the light fake is e.

   Thus, both fakes can be identified with the 3 prescribed weighings.

   Solution 2. Yes, Nastya can choose the weighings so that she can assuredly determine both fake coins and state which of them is heavier. Denote the five coins by a, b, c, d and e. Nastya can achieve the desired outcome by demanding the following three weighings:

(1) a, b vs c, d

(2) a, c vs b, d

(3) a, d vs b, c.

We note that having omitted e from the weighings, there are $\binom{4}{2} = 6$ possible ways of choosing a pair from a, b, c, d. These make up the pairs on the sides of the weighings. We have the following cases.

Case 1: e is fake. Then only one of a, b, c, d is fake, and causes each of the weighings to be unbalanced.
If the light sides of (1), (2) and (3) have a common element, then that common element is the light fake and e is the heavy fake.
If the heavy sides of (1), (2) and (3) have a common element, then that common element is the heavy fake and e is the light fake.

Case 2: The fake coins are among a, b, c, d. Then the fake coins are together on one side in a unique weighing.
Without loss of generality, suppose (1) is balanced. Then either a and b are fakes or c and d are. The common coin on the light sides of (2) and (3) is the light fake coin, and the common coin on the heavy side of (2) and (3) is the fake heavy coin.

Alternative way of distinguishing the cases: For each of the coins a, b, c, d, Nastya determines how often the coin was on the heavy side, e.g. 2220 indicates, in order, that each of a, b, and c were on the heavy side twice, and d was never on the heavy side. The coin e can be a a heavy fake coin, light fake coin or a real coin. The cases are distinguished by:

e is a heavy fake coin if and only if the coding of the three weighings is a permutation of 2220.
e is a light fake coin if and only if the coding of the three weighings is a permutation of 3111.
e is a real coin if and only if the coding of the three weighings is a permutation of 2110.

In the first case 2220, the light fake is the coin corresponding to 0. In the case 3111, the heavy fake is the coin corresponding to 3. Finally, in the case 2110, the heavy fake is the coin corresponding to 2, and the light fake is the coin corresponding to 0.

Thus, both fakes can be identified with the 3 prescribed weighings.

Note. There are no other ways for Nastya to prescribe the weighings to identify two fake coins than the weighings described above.

5. **Solution 1.** Any nine-digit integer N can be represented in the following way

$$N = 10^6 A + 10^3 B + C = 999 \cdot (1001A + B) + (A + B + C),$$

where A, B and C are the numbers formed by the first three digits of N, the middle three digits of N, and the last three digits of N, respectively.

Since $1+2+\cdots+9 = 45$, one can partition the digits 1, 2, ..., 9 into three triples, having a common sum of 15; for example, (1, 5, 9), (2, 6, 7) and (3, 4, 8). If the three digits from one triple are placed in the leftmost positions of the numbers A, B and C, the digits of another triple are placed in the middle positions of A, B and C, and the digits of the third triple are placed in the rightmost positions of A, B and C, then $A + B + C = 15 \cdot 111 = 45 \cdot 37$. Since 37 also divides 999, a beautiful number with such a configuration for A, B and C will be divisible by 37. Since we have six ways to arrange the digits of a triple in the designated position of A, B and C, for each of three triples, and also there are six ways to arrange the triples among the digit positions, we have at least $6^4 = 1296$ beautiful numbers, each of which is divisible by 37.

Solution 2, by William Steinberg. Observe that $3 \cdot 37 = 11$ and $3^3 \cdot 37 = 999$, so that

$$10^2 \equiv -11 \pmod{37} \qquad \text{and} \qquad 10^3 \equiv 1 \pmod{37}.$$

Let the decimal representation of a beautiful number be $\overline{a_8 a_7 \dots a_1 a_0}$. Then

$$\begin{aligned}\overline{a_8 a_7 \dots a_1 a_0} &= \sum_{k=0}^{8} a_k \times 10^k \\ &\equiv (a_0 + a_3 + a_6) \cdot 1 \\ &\quad + (a_1 + a_4 + a_7) \cdot 10 \\ &\quad + (a_2 + a_5 + a_8) \cdot (-11) \pmod{37}\end{aligned}$$

Observe that

$$1 + 10 + (-10) \equiv 0 \pmod{37}.$$

So, if it is possible to have

$$\begin{aligned}a_0 + a_3 + a_6 = a_1 + a_4 + a_7 &= a_2 + a_5 + a_8 \\ &= S, \text{ say,}\end{aligned}$$

then $\overline{a_8 a_7 \dots a_1 a_0} \equiv 0 \pmod{37}$.

Since $1 + 2 + \cdots + 9 = 45$, indeed we can have $S = 15$. Let

$$\begin{aligned}\mathcal{D} &= \{\{a_0, a_3, a_6\},\ \{a_1, a_4, a_7\},\ \{a_2, a_5, a_8\}\} \\ \mathcal{T} &= \{\{1, 6, 8\},\ \{2, 4, 9\},\ \{3, 5, 7\}\}\end{aligned}$$

where $\mathcal{D}$ are the digit triples whose sum we want to be 15, and $\mathcal{T}$ is a partition of $\{1, 2, \dots, 9\}$ into triples whose sum is 15. Then the digit triples in $\mathcal{T}$ can be assigned to the three triples in $\mathcal{D}$ in $3! = 6$ ways, and after such an assignment the digits of each $\mathcal{T}$-triple can be assigned to the digits of a $\mathcal{D}$-triple in $3! = 6$ ways. So there are at least $6^4 = 1296 > 1000$ beautiful numbers that are divisible by 37.

Remark. There are many ways of partitioning $\{1, 2, \dots, 9\}$ into triples whose sum is 15 (so-called *magic square* triples). Solutions 1. and 2. show two ways. And there are also beautiful numbers with 0 as one of their digits that are divisible by 37. A computer search shows the bound can be relaxed from 1000 to over 89 000.